Mathematical Journal of Okayama University

Volume 26, *Issue* 1 1984 *Article* 1

JANUARY 1984

The spectrum of a conjugacy class graph of a finite group

Noboru Ito*

*Konan University

Math. J. Okayama Univ. **26** (1984), 1 —10

THE SPECTRUM OF A CONJUGACY CLASS GRAPH OF A FINITE GROUP

NOBORU ITO

Introduction. Let G be a finite group, C a conjugacy class of G and $h = |C|$, the number of elements in C. Then a conjugacy class graph $\Gamma = \Gamma(G, C)$ is defined as follows. The vertex set $V(\Gamma)$ of Γ is G. For elements x and y in $G, \{x, y\}$ belongs to the edge set $E(\Gamma)$ of Γ if $y = cx$ for some element c in C. In order to secure the undirected and connected property of Γ we assume that $C = C^{-1}$ and $G = \langle C \rangle$, namely that C is real and G is generated by C.

The main purpose of this note is to show that the spectrum of Γ can be determined if the portion of the character table of G corresponding to C and the identity element e of G is known. Namely we prove the following theorem (§ 1).

Theorem. *Let χ be an irreducible character of G. Then χ contributes for the spectrum of Γ an eigenvalue $\lambda = h\chi(c)/\chi(e)$, where c is an element of C, with multiplicity $\chi(e)^2$. Distinct characters may contribute the same eigenvalue. Therefore the multiplicity of λ in the spectrum of Γ equals the sum of all $\chi(e)^2$'s such that $h\chi(c)/\chi(e) = \lambda$.*

In § 2 we consider a condition for Γ to be bipartite, and we see that Γ is bipartite if G is solvable. In § 3 we describe a special situation where G is a symmetric group and C is the class of transpositions. In § 4 we state a few remarks.

1. Proof of the theorem. In order to describe the adjacency matrix A of Γ we need a labelling of elements of G. So we put $G = \{x_1, x_2, \ldots, x_g\}$, where g denotes the order of G. Let $\delta_C(z) = 1$ or 0 according as z is in C or not. Then we can put $A = (\delta_C(x_i x_j^{-1}))$, $1 \leqq i, j \leqq g$.

Now δ_C is a class function of G. Therefore δ_C is a $\mathbb{C}$-linear combination of irreducible characters of G, where $\mathbb{C}$ denotes the field of complex numbers :

$$\delta_C = \sum_{i=1}^{k} a_i \chi_i, \ a_i \in \mathbb{C},$$

Mathematical Journal of Okayama University, Vol. 26 [1984], Iss. 1, Art. 1

where k denotes the number of distinct irreducible characters of G. By the orthogonality relation of irreducible characters of G we obtain that

$$a_l g = \sum_{j=1}^{g} \delta_C(x_j)\chi_l(x_j^{-1}) = \sum_{x\in C} \chi_l(x^{-1}) = h\chi_l(c),\ c \in C = C^{-1}.$$

So we have that $a_l = h\chi_l(c)/g$.

Put $D_l(\chi_l(x_i x_j^{-1}))$, $1 \leqq i, j \leqq g$. Then we have that $A = (h/g)\sum_{l=1}^{k} \chi_l(c)D_l$. Here we recall the following relation of group characters ([2], p. 32):

$$\sum_{i=1}^{g} \chi_s(x_i^{-1})\chi_r(x_i y) = (\chi_r(y)/\chi_r(e))g\delta_{r,s}, \tag{1}$$

where $\delta_{r,s}$ denotes the Kronecker delta.

Put $X_{s,1} = (\chi_s(x_1^{-1}), \ldots, \chi_s(x_g^{-1}))$, $1 \leqq s \leqq k$, and consider $X_{s,1}A = (h/g)\sum_{l=1}^{k} \chi_l(c)X_{s,1}D_l$. Then by (1) the j-th component of $X_{s,1}D_l$ equals $\sum_{i=1}^{g} \chi_s(x_i^{-1})\chi_l(x_i x_j^{-1}) = (\chi_s(x_j^{-1})/\chi_s(e))g\delta_{s,l}$. So the j-th component of $X_{s,1}A$ equals $(h\chi_s(c)/\chi_s(e))\chi_s(x_j^{-1})$. Namely $X_{s,1}$ is an eigenvector of A corresponding to the eigenvalue $h\chi_s(c)/\chi_s(e)$.

Now we recognize that $X_{s,1}$ is the first row vector of D_s. Let $X_{s,m}$ be the m-th row vector of D_s, $2 \leqq m \leqq g$. Then $X_{s,m}$ is an eigenvector of A corresponding to the eigenvalue $h\chi_s(c)/\chi_s(e)$, too. In fact, we notice that

$$\sum_{i=1}^{g} \chi_s(x_m x_i^{-1})\chi_l(x_i x_j^{-1}) = \sum_{i=1}^{g} \chi_s(x_i^{-1})\chi_l(x_i x_m x_j^{-1}) = (g\chi_s(x_m x_j^{-1})/\chi_s(e))d_{s,l}.$$

Then any linear combination of the $X_{s,m}$, $1 \leqq m \leqq g$, is an eigenvector of A corresponding to the eigenvalue $h\chi_s(c)/\chi_s(e)$.

As remarked in the formulation of the theorem it is possible that $\chi_s(c)/\chi_s(e) = \chi_t(c)/\chi_t(e)$ for $s \neq t$. However, for $s \neq t$, X_{sl} and X_{tj} are orthogonal as complex vectors. Namely by (1) we have that $X_{sl}\cdot X_{tm} = \sum_{i=1}^{g} \chi_s(x_l x_i^{-1})\chi_t(x_i x_m^{-1}) = \sum_{i=1}^{g} \chi_s(x_i^{-1})\chi_t(x_i x_l x_m^{-1}) = 0$. Thus in order to complete the proof of the theorem it remains to show that the rank af D_l equals $\chi_l(e)^2$, $1 \leqq l \leqq k$.

Let R be the regular representation of G and γ the character of R. So $\gamma(z) = g$ or 0 according as $z = e$ or not. Since $\gamma = \sum_{l=1}^{k} \chi_l(e)\chi_l$ ([2],

p. 14), we have that

$$(\gamma(x_i x_j^{-1})) = \sum_{l=1}^{k} \chi_l(e) D_l. \tag{2}$$

Let e_i be the standard basis vector of $\mathbb{C}_g$, the space of all complex row vectors of size g, $1 \leqq i \leqq g$. Then (2) yields that $ge_i = \sum_{l=1}^{k} \chi_l(e) X_{li}$, $1 \leqq i \leqq g$. Namely the $X_{s,i}$, $1 \leqq s \leqq k$, $1 \leqq i \leqq g$, generates $\mathbb{C}_g$. Therefore, since $g = \sum_{l=1}^{k} \chi_l(e)^2$ ([2], p. 14), it suffices to show that the rank of D_l does not exceed $\chi_l(e)^2$, $1 \leqq i \leqq k$.

Let $R_l(x) = (a_{rs}^{(l)}(x))$, $1 \leqq r$, $s \leqq \chi_l(e)$, $x \in G$ be an irreducible representation of G corresponding to the character χ_l, $1 \leqq l \leqq k$. Then it holds that $R_l(x_i) R_l(x_j^{-1}) = R_l(x_i x_j^{-1})$. Thus we obtain that

$$\chi_l(x_i x_j^{-1}) = \sum_{r,t=1}^{\chi_l(e)} a_{rt}^{(l)}(x_i) a_{tr}^{(l)}(x_j^{-1}). \tag{3}$$

Let $A_i^{(l)} = \begin{pmatrix} a_{i1}^{(l)}(x_1), & \dots, & a_{i\chi_l(e)}^{(l)}(x_1) \\ \vdots & & \vdots \\ a_{i1}^{(l)}(x_g), & \dots, & a_{i\chi_l(e)}^{(l)}(x_g) \end{pmatrix} \begin{pmatrix} a_{1i}^{(l)}(x_1^{-1}), & \dots, & a_{1i}^{(l)}(x_g^{-1}) \\ \vdots & & \vdots \\ a_{\chi_l(e)i}^{(l)}(x_1^{-1}), & \dots, & a_{\chi_l(e)i}^{(l)}(x_g^{-1}) \end{pmatrix}$, $1 \leqq i \leqq \chi_l(e)$.

Then by (3) we have that $D_l = A_1^{(l)} + \cdots + A_{\chi_l(e)}^{(l)}$. Obviously the rank of each $A_i^{(l)}$ does not exceed $\chi_l(e)$. Therefore the rank of D_l does not exceed $\chi_l(e)^2$.

This completes the proof.

Remark. Since our eigenvectors are independent from the choice of C, a similar theorem holds, when C is replaced by an inverse closed union of conjugacy classes.

2. Bipartition condition.

Proposition 1. *Let G' denote the commutator subgroup of G. Then $\Gamma(G, C)$ is bipartite if and only if G/G' has order two and $G = G'\langle c \rangle$, $c \in C$.*

Proof. Suppose that G/G' has order two and $G = G'\langle e \rangle$, $c \in C$. Now $h = |C|$ is the largest eigenvalue of $\Gamma(G, C)$. Let η be the linear character of G whose kernel equals G'. Then η yields the eigenvalue $-h$. Therefore by a result of A.J. Hoffman ([4], p. 227) $\Gamma(G, C)$ is bipartite.

Suppose that $\Gamma(G, C)$ is bipartite. Let N be the set of all elements x of G such that the distance $d(e, x)$ from e is even. Then since $\Gamma(G, C)$ contains no odd cycle ([1], p. 50), N forms a subgroup of G of index two. If G/G' has order larger than two, then we have that $G \neq \langle C \rangle$.

Proposition 2. *If G is solvable, then $\Gamma(G, C)$ is bipartite.*

Proof. Let N be a normal maximal subgroup of prime index p. If $p \neq 2$, then $C \neq C^{-1}$. Thus we get $p = 2$. By the same reason we have that $N = G'$.

3. A special case. Let $G =$ Sym n be a symmetric group on n letters and C the class of transpositions. Then $h = |C| = n(n-1)/2$. Let $\Gamma(n) = \Gamma(G, C)$. Then eigenvalues of $\Gamma(n)$ are given explicitly in term of the characteristics of Young diagrams. However, we have to recognize that, for a given n the number of Young diagrams equals $p(n)$, the number of partitions of n, and $p(n)$ increases very rapidly when n increases.

Let χ be an irreducible character of G corresponding to the Young diagram $Y(\chi)$. Now the number r of nodes on the diagonal initiating at the top-left of $Y(\chi)$ is called the rank of $Y(\chi)$. Let a_i be the number of nodes to the right of the i-th node on the diagonal and b_i the number of nodes beneath the i-th node on the diagonal, $1 \leqq i \leqq r$. Then $\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$ is called the characteristic of $Y(\chi)$. For example, in the Young diagram

$$\begin{array}{llll} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & & & \end{array}$$

we have that $r = 2$, $a_1 = 3$, $a_2 = 1$, $b_1 = 4$ and $b_2 = 2$. In general, we have that $a_1 > a_2 > \cdots > a_r \geqq 0$, $b_1 > b_2 > \cdots > b_r \geqq 0$ and $n - r = \sum_{i=1}^{r} (a_i + b_i)$. Now we have a result of G. Frobenius ([3a, b]):

$$h\chi(c)/\chi(e) = (1/2)\left(\sum_{i=1}^{r} b_i(b_i+1) - \sum_{i=1}^{r} a_i(a_i+1)\right).$$

It might be interesting to have a detailed description of the spectrum of $\Gamma(n)$. We prove the following proposition.

Proposition 3. *If n is not so small, then every integer i in the interval*

$[-n, n]$ *is an eigenvalue of* $\Gamma(n)$.

Remark. They are yielded by characters of rank at most three. Our bounds for n are probably not best possible.

Proof. We identify the character with the characteristic of the corresponding Young diagram. Since $\Gamma(n)$ is bipartite (see Proposition 1), we may assume that $0 \leqq i \leqq n$. It is convenient to divide into four cases.

(i) The case $n \equiv 1 \pmod 4$. Let $n = 4m+1$.

The characters $\begin{pmatrix} 2m \\ 2m \end{pmatrix}$ and $\begin{pmatrix} 2m-1 \\ 2m+1 \end{pmatrix}$ yield the eigenvalues 0 and n respectively.

$\begin{pmatrix} x & y \\ x & y+1 \end{pmatrix}$ yields the eigenvalue $i = y+1$. Since $2x+2y = n-3$, $0 \leqq y$ and $y+2 \leqq x$, we get all values i such that $1 \leqq i \leqq (n-5)/4$. Here the bound for n is 9.

$\begin{pmatrix} x & y \\ x+1 & y \end{pmatrix}$ yields the eigenvalue $i = x+1 = (n-1)/2-y$. Since $2x+2y = n-3$, $0 \leqq y$ and $y+1 \leqq x$, we get all values i such that $(n+3)/4 \leqq i \leqq (n-1)/2$. Here the bound for n is 5.

$\begin{pmatrix} x & y \\ x+1 & y+2 \end{pmatrix}$ yields the eigenvalue $i = x+2y+4 = y+(n+3)/2$. Since $2x+2y = n-5$, $0 \leqq y$ and $y+2 \leqq x$, we get all values i such that $(n+3)/2 \leqq i \leqq 3(n-1)/4$. Here the bound for n is 9.

$\begin{pmatrix} x & y \\ x+2 & y+1 \end{pmatrix}$ yields the eigenvalue $i = 2x+y+4 = n-1-y$. Since $2x+2y = n-5$, $0 \leqq y$ and $y+1 \leqq x$, we get all values i such that $(3n+5)/4 \leqq i \leqq n-1$. Here the bound for n is 9.
So there are only three gaps, namely $(n-1)/4$, $(n+1)/2$ and $(3n+1)/4$. We fill these three gaps by using characters of rank three.

$\begin{pmatrix} (n-1)/4 & (n-9)/4 & 0 \\ (n-1)/4 & (n-5)/4 & 1 \end{pmatrix}$ yields the eigenvalue $(n-1)/4$. Here the bound for n is 13.

$\begin{pmatrix} (n+7)/4 & (n-25)/4 & 0 \\ (n+7)/4 & (n-17)/4 & 4 \end{pmatrix}$ yields the eigenvalue $(n+1)/2$. Here the bound for n is 37.

Let $n \equiv 1 \pmod 8$. Then $\begin{pmatrix} (3n-11)/8 & (n-9)/8 & 0 \\ (3n+5)/8 & (n-9)/8 & 0 \end{pmatrix}$ yields the eigenvalue $(3n+1)/4$. Here the bound for n is 17.

Mathematical Journal of Okayama University, Vol. 26 [1984], Iss. 1, Art. 1

Let $n \equiv 5 \pmod 8$. Then $\begin{pmatrix} (3n-23)/8 & (n-5)/8 & 0 \\ (3n-7)/8 & (n-5)/8 & 2 \end{pmatrix}$ yields the eigenvalue $(3n+1)/4$. Here the bound for n is 29.

(ii) The case $n \equiv 3 \pmod 4$.

We may proceed as in the case $n \equiv 1 \pmod 4$. Bounds for n in this case are 7, 7, 11 and 7 instead of 9, 5, 9 and 9. We get three gaps again. They are $(n+1)/4$, $(n+1)/2$ and $(3n-1)/4$.

$\begin{pmatrix} (n+13)/4 & (n-27)/4 & 0 \\ (n+13)/4 & (n-23)/4 & 3 \end{pmatrix}$ yields the eigenvalue $(n+1)/4$. Here the bound for n is 39.

$\begin{pmatrix} (n+25)/4 & (n-47)/4 & 0 \\ (n+25)/4 & (n-39)/4 & 6 \end{pmatrix}$ yields the eigenvalue $(n+1)/2$. Here the bound for n is 67.

Let $n \equiv 3 \pmod 8$. Then $\begin{pmatrix} (3n-25)/8 & (n-3)/8 & 0 \\ (3n-9)/8 & (n-3)/8 & 2 \end{pmatrix}$ yields the eigenvalue $(3n-1)/4$. Here the bound for n is 27.

Let $n \equiv 7 \pmod 8$. Then $\begin{pmatrix} (3n-13)/8 & (n-7)/8 & 0 \\ (3n+3)/8 & (n-7)/8 & 0 \end{pmatrix}$ yields the eigenvalue $(3n-1)/4$. Here the bound for n is 15.

(iii) The case $n \equiv 0 \pmod 4$.

$\begin{pmatrix} x & y & 0 \\ x & y+1 & 0 \end{pmatrix}$ yields the eigenvalue $i = y+1$. Since $2x+2y = n-4$, $1 \leqq y$ and $y+2 \leqq x$, we get all values i such that $2 \leqq i \leqq (n-4)/4$. Here the bound for n is 12.

$\begin{pmatrix} x & y & 0 \\ x+1 & y & 0 \end{pmatrix}$ yields the eigenvalue $i = x+1 = (n-2)/2-y$. Since $2x+2y = n-4$, $1 \leqq y$ and $y+1 \leqq x$, we get all values i such that $(n+4)/4 \leqq i \leqq (n-4)/2$. Here the bound for n is 12.

$\begin{pmatrix} x & y & 0 \\ x+1 & y+2 & 0 \end{pmatrix}$ yields the eigenvalue $i = x+2y+4 = (n+2)/2+y$. Since $2x+2y = n-6$, $1 \leqq y$ and $y+2 \leqq x$, we get all values i such that $(n+4)/2 \leqq i \leqq (3n-8)/4$. Here the bound for n is 16.

$\begin{pmatrix} x & y & 0 \\ x+2 & y+1 & 0 \end{pmatrix}$ yields the eigenvalue $i = 2x+y+4 = n-2-y$. Since $2x+2y = n-6$, $1 \leqq y$ and $y+1 \leqq x$, we get all values i such that $(3n)/4 \leqq i \leqq n-3$. Here the bound for n is 12.

So there are ten gaps, namely 0, 1, $n/4$, $(n-2)/2$, $n/2$, $(n+2)/2$, $(3n-4)/4$, $n-2$, $n-1$ and n.

$\begin{pmatrix}(n-2)/2 & 0\\ (n-2)/2 & 0\end{pmatrix}$ yields the eigenvalue 0. Here the bound for n is 4.

$\begin{pmatrix}(n-8)/2 & 2 & 0\\ (n-8)/2 & 2 & 1\end{pmatrix}$ yields the eigenvalue 1. Here the bound for n is 16.

Let $n\equiv 0 \pmod 8$. Then $\begin{pmatrix}(3n-8)/8 & n/8\\ 3n/8 & (n-8)/8\end{pmatrix}$ yields the eigenvalue $n/4$. Here the bound for n is 8. Let $n\equiv 4 \pmod 8$. Then $\begin{pmatrix}(3n-4)/8 & (n-12)/8\\ (3n-4)/8 & (n+4)/8\end{pmatrix}$ yields the eigenvalue $n/4$. Here the bound for n is 12.

$\begin{pmatrix}(n+4)/4 & (n-20)/4 & 0\\ (n+4)/4 & (n-12)/4 & 3\end{pmatrix}$ yields the eigenvalue $(n-2)/2$. Here the bound for n is 28. $\begin{pmatrix}(n/2)-1\\ n/2\end{pmatrix}$ yields the eigenvalue $n/2$. $\begin{pmatrix}n/4 & (n-16)/4 & 0\\ n/4 & (n-8)/4 & 3\end{pmatrix}$ yields the eigenvalue $(n+2)/2$. Here the bound for n is 24. Let $n\equiv 0$ (mod 8). Then $\begin{pmatrix}(3n-40)/8 & (n+8)/8 & 0\\ (3n-24)/8 & (n+8)/8 & 3\end{pmatrix}$ yields the eigenvalue $(3n-4)/4$. Here the bound for n is 32. Let $n\equiv 4 \pmod 8$. Then $\begin{pmatrix}(3n-20)/8 & (n-4)/8 & 0\\ (3n-4)/8 & (n-4)/8 & 1\end{pmatrix}$ yields the eigenvalue $(3n-4)/4$. Here the bound for n is 20. $\begin{pmatrix}n/4 & (n-12)/4\\ (n+4)/4 & n/4\end{pmatrix}$ yields the eigenvalue $n-1$. Here the bound for n is 12.

$\begin{pmatrix}(n-4)/2 & 0\\ n/2 & 0\end{pmatrix}$ yields the eigenvalue $n-1$. Here the bound for n is 8.

$\begin{pmatrix}(n/4)-1 & (n/4)-2\\ (n/4)+1 & n/4\end{pmatrix}$ yields the eigenvalue n. Here the bound for n is 8.

(iv) The case $n\equiv 2 \pmod 4$.

We may proceed as in the case $n=0 \pmod 4$. Bounds for n in this case are 14, 10, 14 and 14 instead of 12, 12, 16 and 12. We get ten gaps again. They are 0, 1, $(n-2)/4$, $(n-2)/2$, $n/2$, $(n+2)/2$, $(3n-2)/4$, $n-2$, $n-1$ and n. Moreover, for $i=0$, 1, $n/2$ and $n-1$ we may proceed as in the case $n\equiv 0 \pmod 4$. For $i=0$, 1 and $n-1$ bounds for n in this case are 6, 14 and 6 instead of 4, 16 and 8.

$\begin{pmatrix}(n+6)/4 & (n-18)/4 & 0\\ (n+6)/4 & (n-14)/4 & 2\end{pmatrix}$ yields the eigenvalue $(n-2)/4$. Here the bound for n is 26. Let $n\equiv 2 \pmod 8$. Then $\begin{pmatrix}(3n-14)/8 & (n+6)/8\\ (3n+2)/8 & (n-10)/8\end{pmatrix}$

yields the eigenvalue $(n-2)/2$. Here the bound for n is 18. Let $n\equiv 6 \pmod 8$. Then $\begin{pmatrix}(3n-2)/8 & (n-22)/8\\ (3n-2)/8 & (n+10)/8\end{pmatrix}$ yields the eigenvalue $(n-2)/2$. Here the bound for n is 22. Let $n\equiv 2 \pmod 8$. Then $\begin{pmatrix}(3n-6)/8 & (n-18)/8\\ (3n-6)/8 & (n+14)/8\end{pmatrix}$ yields the eigenvalue $(n+2)/2$. Here the bound for n is 18. Let $n\equiv 6 \pmod 8$. Then $\begin{pmatrix}(3n-10)/8 & (n+2)/8\\ (3n+6)/8 & (n-14)/8\end{pmatrix}$ yields the eigenvalue $(n+2)/2$. Here the bound for n is 14. Let $n\equiv 2 \pmod 8$. Then $\begin{pmatrix}(3n-38)/8 & (n+6)/8 & 0\\ (3n-22)/8 & (n+6)/8 & 3\end{pmatrix}$ yields the eigenvalue $(3n-2)/4$. Here the bound for n is 26. Let $n\equiv 6 \pmod 8$. Then $\begin{pmatrix}(3n-18)/8 & (n-6)/8 & 0\\ (3n-2)/8 & (n-6)/8 & 1\end{pmatrix}$ yields the eigenvalue $(3n-2)/4$. Here the bound for n is 22. $\begin{pmatrix}(n-50)/2 & 18 & 0\\ (n-46)/2 & 18 & 9\end{pmatrix}$ yields the eigenvalue $n-2$. Here the bound for n is 90. Finally $\begin{pmatrix}(n-2)/4 & (n-10)/4\\ (n+6)/4 & (n-2)/4\end{pmatrix}$ yields the eigenvalue n. Here the bound for n is 10.

We add a proposition which states a well known fact on G in a graph theoretical terminology.

Proposition 4. *Let x be an element of G whose cycle structure consists of cycles of lengths $n_1, n_2, \ldots, n_r$, where $n = n_1+n_2+\cdots+n_r$. Then the distance $\mathrm{d}(e,x)$ from e to x in $\Gamma(n)$ equals $n-r$.*

Proof. Assume that there exists an x such that x is a product of m transpositions where m is less than $n-r$. Then choose x so that m is the least. Let $x=(a_1b_1)\cdots(a_mb_m)$. Then we may assume that either the first cycle of the cycle structure of x is of the form $(a_1 \ldots b_1 \ldots)$ or the first and second cycles are of the form $(a_1 \ldots)(b_1 \ldots)$. Consider $(a_1b_1)x$. Then it is a product of $m-1$ transpositions. On the other hand, the cycle structure of $(a_1b_1)x$ consists of cycles of lengths either $n_{11}, n_{12}, n_2, \ldots, n_r$, where $n_1 = n_{11}+n_{12}$, or $n_1+n_2, n_3, \ldots, n_r$. In both cases it contradicts the least property of m.

In particular, the diameter of $\Gamma(n)$ equals $n-1$.

4. Remarks. Let $\Gamma=\Gamma(G,C)$ is a conjugacy class graph, and $\mathrm{Aut}\Gamma$ the automorphism group of Γ. Clearly $\mathrm{Aut}\Gamma$ contains G which acts on $V(\Gamma)$

as the right multiplication. Furthermore, AutΓ contains $G/Z(G)$, where $Z(G)$ denotes the center of G. In this case G acts on $V(\Gamma)$ as the conjugation and so $Z(G)$ is the kernel of the action. Hence Γ is symmetric. Now we notice the following.

Proposition 5. *The mapping σ on $V(\Gamma)$ defined by $x\sigma = x^{-1}$, $x \in G$, belongs to* AutΓ.

Proof. If $y = cx$, $c \in C$, $x, y \in G$, then $y^{-1} = x^{-1}c^{-1} = x^{-1}c^{-1}xx^{-1}$.

If $G = \langle C \rangle$, $C \neq C^{-1}$ and $\Gamma = \Gamma(G, C \cup C^{-1})$, then Proposition 5 shows that Γ is also symmetric. It may be interesting to determine AutΓ.

Proposition 6. *The girth of $\Gamma = \Gamma(G, C)$ is at most four.*

Proof. Let x and y be distinct elements of C. Then the sequence 1, x, $xy = xyx^{-1} \cdot x$, y forms a cycle.

A conjugacy class C of a finite group G is called rational, if an element c of C has order r and if s is relatively prime to r, then c^s belongs to C. So any class of involutions is rational.

Proposition 7. *A conjugacy class graph $\Gamma = \Gamma(G, C)$ is integral if and only if C is rational.*

Proof. It is well known that $h\chi(c)/\chi(e)$ is an algebraic integer for every irreducible character χ of G ([2]). Now let C be rational. $\chi(c)$ is a sum of some r-th roots of unity and any algebraic conjugate of $\chi(c)$ equals to $\chi(c^s)$ for some integer s prime to r. Since $\chi(c) = \chi(c^s)$ by the definition of C, $\chi(c)$ is a rational integer. Thus $h\chi(c)/\chi(e)$ is a rational integer, and hence Γ is integral. Conversely if $h\chi(c)/\chi(e)$ is a rational integer for every irreducible character χ, then $\chi(c)$ is rational for every irreducible character χ. If s is relatively prime to r, then $\chi(c)$ and $\chi(c^s)$ are algebraically conjugate. So $\chi(c) = \chi(c^s)$ for every irreducible character χ. By the orthogonality relation of group characters this implies that c and c^s are conjugate.

For integral graphs see ([5]).

References

[1] N. Biggs: Algebraic graph theory, Cambridge 1974.

[2] W. FEIT : Characters of finite groups, Benjamin 1967.
[3a] G. FROBENIUS : Ueber die Charaktere der symmetrischen Gruppe, Berliner Sitzungsberichte 1900, 516–534.
[3b] G. FROBENIUS : Ueber die Charaktere der mehrfach transitiven Gruppen, Berliner Sitzungsberichte 1904, 528–571.
[4] D.R. FULKERSON : Studies in graph theory II, MAA 1975.
[5] F. HARARY and A.J. SCHWENK : Which graphs have integral spectra ?, LNM 406, Springer 1974, 45–51.

DEPARTMENT OF APPLIED MATHEMATICS
KONAN UNIVERSITY
KOBE, 658

(*Received April 1, 1984*)